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NONEXISTENCE OF RATIONAL ROTATION MINIMIZING FRAMES ON QUINTIC HELICES

Fatma Şengüler-Çiftçi
Johann Bernoulli Institute for
Mathematics and Computer Science
University of Groningen
Groningen, The Netherlands
email: F.A.Senguler-Ciftci@rug.nl

Gert Vegter
Johann Bernoulli Institute for
Mathematics and Computer Science
University of Groningen
Groningen, The Netherlands
email: G.Vegter@rug.nl

ABSTRACT
Rotation minimizing frames (RMFs) have the property of minimum twist which makes them useful for computer graphics, swept surface or generalised cylinder constructions, motion design and control in computer animation and robotics, streamline visualization, and tool path planning in CAD/CAM. Recent advances in two main research topics, polynomial helices and rational rotation minimizing frame (RRMF) curves, in the theory of Pythagorean-hodograph (PH) curves let us study RRMFs on quintic helices. We first give a condition on a polynomial helix of any degree to have an RRMF, which leads a simplification of rational approximation to RMFs on monotone-helical PH quintics. Then we prove the nonexistence of RRMFs on both general and monotone quintic helices.

KEY WORDS
Quintic helices, rotation minimizing frames.

1 Introduction

A parametric curve is called a Pythagorean-hodograph (PH) curve if it has a polynomial speed at each instant. The theory of PH curves is a much studied research topic in Computer Aided Geometric Design (CAGD) because of their useful properties. Recent developments in this topic are documented in the book [1], for instance. Polynomial helices and rational rotation minimizing frame curves are two important subclasses of PH curves. Although both topics have been studied by many authors so far there is not much work which combines the two. The aim of the present paper is to start a line of research in this direction by giving a special emphasis to helical PH quintic curves.

A polynomial helix is a PH curve if its curvature as being a polynomial equal to its torsion, which is also a polynomial, times a constant. On the other hand, it is known that a cubic PH curve is a helical curve [1]. Nevertheless, quintic curves form a larger class than their subfamily of quintic helices. We analyse two kinds of PH quintic helices to have a rational rotation minimizing frame (RRMF): the first type is the class of monotone-helical PH quintics, whose hodographs have components with a common quadratic factor; the second type is the class of general helical PH quintics, whose hodographs are irreducible.

For background, the reader may like to look at some references from the literature. For example, in [2], characterizations of quintic helical curves are studied in detail. A helicity condition of polynomial helices of any degree is then given from a geometric point of view in [3]. For a more detailed study on quintic and degree seven polynomial helices and double PH curves we refer to [4, 5].

The rotation-minimizing frames (RMFs) are proved to be useful in animation, robotics path planning and control, modeling of swept surfaces or generalised cylinders, and manufacturing applications [6], [7]. To illustrate minimum twist of an RMF, we provide Figures 1 and 2 which give a rotation minimizing frame vector and the binormal vector of a PH quintic curve. Especially *rational* RMFs are useful for computational purposes. In practice one often looks for curves with an adapted frame having minimal rotation. In [8], it is shown that cubic PH curves do not have RRMFs. In the same work, a general condition for a PH curve to have an RRMF is also given. Quintic RRMF curves are identified in [9], then more symmetric characterizations of these curves are given in [10]. RRMF curves of arbitrary degree are studied in another recent work [11].

As we mentioned before, the study of helical PH curves and the study of rational minimizing frames are quite well developed but these two subjects together have not been explored too much. The only nonexistence of cubic PH curves is given in [8]. By the aid of recent developments in helical quintics [2, 12], and quintic RRMF curves [9, 10] we are equipped to study the conditions of the existence of RRMF on helices. Our observations give us the results on the nonexistence of RRMFs on quintic helices.

The present paper is organized as follows. Section 2 introduces definitions of and basic results on PH curves, PH helices, RMF and RRMF curves in quaternion and Hopf map representations. In Section 3 we give a necessary and sufficient condition for helices of any degree to have RRMFs. Section 4 is devoted to the main results of this paper, where the case of quintic helical curves have been throughly ex-

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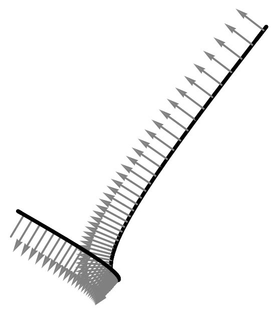

Figure 1. A Rotation minimizing frame of a PH quintic curve. Only one of the normal vectors is shown.

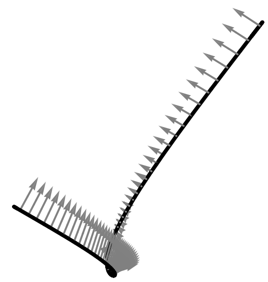

Figure 2. The Frenet-Serret frame of the same curve with Figure 1. Only binormal vector is shown.

amined resulting in the nonexistence of such frames. Subsequently in Section 5 we draw some conclusions.

2 Background

In this section we review the preliminary material which we use in the main part of the paper.

2.1 Quaternion and Hopf Map Form of PH Curves

A polynomial curve $\mathbf{r}(t)$ is called a *Pythagorean hodograph* (PH) curve if its hodograph $\mathbf{r}'(t)$ has a polynomial norm. Namely, if in coordinates

$$\mathbf{r}(t) = (x(t), y(t), z(t)), \tag{1}$$

then

$$|\mathbf{r}'(t)| = \sqrt{(x'(t)^2 + y'(t)^2 + z'(t)^2} = \sigma(t) \tag{2}$$

for some polynomial $\sigma(t)$. This implies that there exist four polynomials $u(t), v(t), p(t)$ and $q(t)$ such that

$$\begin{aligned} x'(t) &= u(t)^2 + v(t)^2 - p(t)^2 - q(t)^2, \\ y'(t) &= 2\,[u(t)\,q(t) + v(t)\,p(t)], \\ z'(t) &= 2\,[v(t)\,q(t) - u(t)\,p(t)], \end{aligned} \tag{3}$$

with the parametric speed given by

$$\sigma(t) = u(t)^2 + v(t)^2 + p(t)^2 + q(t)^2. \tag{4}$$

We will use both the quaternion and Hopf map representations of PH curves which we recall here. A quaternion is a linear combination of four basis elements $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ which satisfy the product rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \tag{5}$$

and "1" is the usual real unit. Then, non commutativity of the quaternion product follows from the equations

$$\begin{aligned} &\mathbf{i}\,\mathbf{j} = -\mathbf{j}\,\mathbf{i} = \mathbf{k}, \quad \mathbf{j}\,\mathbf{k} = -\mathbf{k}\,\mathbf{j} = \mathbf{i}, \\ &\mathbf{k}\,\mathbf{i} = -\mathbf{i}\,\mathbf{k} = \mathbf{j}. \end{aligned} \tag{6}$$

In quaternion terms $\mathbf{r}'(t)$ can be represented by the product

$$\mathbf{r}'(t) = \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t), \tag{7}$$

for the quaternion polynomial

$$\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}, \tag{8}$$

and $\mathcal{A}^*(t)$ denotes the quaternionic conjugate of $\mathcal{A}(t)$. From [13] we know that the spatial PH curve (3) can be generated from two complex polynomials through the Hopf map

$$H : \mathbb{C} \times \mathbb{C} \to \mathbb{R}^3,$$

defined by $H(\alpha, \beta) = (2\,\alpha, \bar{\beta}, |\alpha|^2 - |\beta|^2)$. The Hopf map representation of $\mathbf{r}'(t)$ is then given by the expression

$$\mathbf{r}' = (|\alpha|^2 - |\beta|^2, 2\,\mathrm{Re}(\alpha\,\bar{\beta}), 2\,\mathrm{Im}(\alpha\,\bar{\beta})). \tag{9}$$

The complex polynomials $\alpha(t), \beta(t)$ and real polynomials $u(t), v(t), q(t), p(t)$ have the relations

$$\begin{aligned} \alpha(t) &= u(t) + v(t)\,\mathrm{i}, \\ \beta(t) &= q(t) + p(t)\,\mathrm{i}. \end{aligned} \tag{10}$$

This can be seen by identifying the unit basis vector $\mathbf{i}$ and the complex unit i [1]. Therefore, the quaternionic curve given in (8) becomes

$$\mathcal{A}(t) = \alpha(t) + \beta(t)\,\mathbf{k}. \tag{11}$$

A quintic PH curve is a PH curve of degree five, so in both quaternion and Hopf map representations one uses quadratic quaternion curves and complex curves respectively. If the polynomials $x'(t), y'(t), z'(t)$ and the parametric speed $\sigma(t)$ share a common linear factor, then $\mathbf{r}(t)$ is called a *monotone-helical PH quintic* [2]. If $\gcd(x'(t), y'(t), z'(t)) =$ constant, then $\mathbf{r}(t)$ is called a *general helical PH quintic* [2]. A quintic PH curve $\mathbf{r}(t)$ can be expressed in terms of quadratic quaternion curves by

$$\mathcal{A}(t) = (1-t)^2\mathcal{A}_0 + 2(1-t)\,t\mathcal{A}_1 + t^2\mathcal{A}_2, \qquad (12)$$

for three constant quaternions $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$ [10]. Or, in Hopf map form (9) by two quadratic complex polynomials

$$\begin{aligned} \alpha(t) &= (1-t)^2\alpha_0 + 2\,(1-t)t\,\alpha_1 + t^2\alpha_2,\\ \beta(t) &= (1-t)^2\beta_0 + 2\,(1-t)t\,\beta_1 + t^2\beta_2. \end{aligned} \qquad (13)$$

These forms are very useful in formulating quintic helices [2] and RRMF quintic curves [1]. We make use of these forms (12) and (13) to prove our results.

2.2 PH Helices

Helical curves form an important class of PH curves. A helix is a curve whose unit tangent makes a constant angle with a constant vector $\mathbf{u}$. So, $\mathbf{r}(t)$ is a helix if and only if

$$\mathbf{r}'(t)\,\cdot\mathbf{u} = |\mathbf{r}'(t)|\,\cos\psi, \qquad (14)$$

with ψ constant. Recall that this definition is equivalent to $\kappa/\tau =$ constant [14], where κ and τ are the curvature and torsion of the curve respectively. It is known that, a helical polynomial curve is a PH curve, whereas converse is not, in general, true [1]. Cubic PH curves are helices but, for example, helical PH quintic space curves form a subset of all spatial PH quintics.

A geometric characterization of helical PH curves of any degree is given by Monterde [3] as follows: a PH curve $r(t)$ is a helical curve if and only if $\alpha(t)/\beta(t)$ traces a circle or a straight line, here $\alpha(t)$ and $\beta(t)$ are complex polynomials as in the Hopf map representation (cf. (10)). For quintic helices there are particular results which we recall below.

The following result will be used to prove the nonexistence of RRMFs on general helical PH quintic curves.

Theorem 1 ([2])**.** *A sufficient condition for the hodograph (7) to yield a helical PH quintic is that the quaternions $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ in (12) are linearly dependent.*

As pointed out in [2], if $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$ are linearly dependent, namely,

$$\mathcal{A}_1 \;=\; c_0\,\mathcal{A}_0 \;+\; c_2\,\mathcal{A}_2 \qquad (15)$$

for some $c_0, c_2 \in \mathbb{R}$, then $\mathcal{A}(t)$ is a straight line. Also note that (15) is not satisfied, in general, by monotone-helical PH curves [2].

If $\mathbf{r}(t)$ is a monotone-helical PH curve then $\alpha(t) = h(t)\,a(t)$ and $\beta(t) = h(t)\,b(t)$ for linear complex polynomials $a(t), b(t)$ and $h(t)$. Then PH helical quintic curve is given by

$$\mathbf{r}'(t) = |h(t)|^2(2a(t)\bar{b}(t), |a(t)|^2 - |b(t)|^2). \qquad (16)$$

with the identification $\mathbb{R}^3 \simeq \mathbb{C}\times\mathbb{R}$. The following property of a regular monotone-helical quintic curve will be of importance; regular means that $\mathbf{r}'(t) \neq 0$ for each t.

Theorem 2 ([12])**.** *Let $\mathbf{r}(t)$ be a regular monotone-helical quintic curve such that $\mathbf{r}'(t)/|\mathbf{r}'(t)| = (1,0,0)$. Then there exists a unique triple of linear polynomials $a(t)$, $b(t) = t$ and $h(t)$ of (16) such that $h(0) > 0$ and Im$(h'(0)) > 0$.*

The proofs of Theorem 1 and Theorem 2 can be found in the cited references.

2.3 Rotation Minimizing Frames

An *adapted frame* on a space curve $\mathbf{r}(t)$ is an orthonormal basis for $\mathbb{R}^3$ such that, at each curve point, the unit tangent $\mathbf{t} = \mathbf{r}'(t)/\sigma$ with speed $\sigma = |\mathbf{r}'(t)|$, is one basis vector, and the other two basis vectors span the normal plane. The most canonical one is the *Serret-Frenet frame* $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ given by [15],

$$\begin{aligned} \mathbf{t} &= \frac{\mathbf{r}'(t)}{\sigma},\\ \mathbf{b} &= \frac{\mathbf{r}'(t)\times\mathbf{r}''(t)}{|\mathbf{r}'(t)\times\mathbf{r}''(t)|},\\ \mathbf{n} &= \mathbf{b}\times\mathbf{t}. \end{aligned} \qquad (17)$$

Then Serret-Frenet equations giving the change of the Serret-Frenet frame read

$$\begin{aligned} \mathbf{t}' &= \quad \sigma\,\kappa\,\mathbf{n},\\ \mathbf{n}' &= -\,\sigma\,\kappa\,\mathbf{t} \;+\; \sigma\,\tau\,\mathbf{b},\\ \mathbf{b}' &= -\,\sigma\,\tau\,\mathbf{n}, \end{aligned} \qquad (18)$$

where κ is the curvature and τ is the torsion given by the expressions

$$\begin{aligned} \kappa &= \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{\sigma^3},\\ \tau &= \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'(t)\times\mathbf{r}''(t)|^2}. \end{aligned} \qquad (19)$$

There are many other adapted frames associated with a given space curve $\mathbf{r}(t)$, and among them the rotation minimizing frames are the ones which minimize the amount of rotation along the curve. Before recalling the definition we first give the following. The variation of a frame $\{\mathbf{t}, \mathbf{f}_1, \mathbf{f}_2\}$ defined on a curve $\mathbf{r}(t)$ is given by its vector angular velocity ω with the relations

$$\mathbf{t}' = \omega\times\mathbf{t}, \quad \mathbf{f}_1' = \omega\times\mathbf{f}_1, \quad \mathbf{f}_2' = \omega\times\mathbf{f}_2. \qquad (20)$$

The magnitude and direction of ω specify the frame angular speed and rotation axis of the frame $\{\mathbf{t}, \mathbf{f}_1, \mathbf{f}_2\}$ [9]. We can write the angular velocity as linear combination of the frame vectors:

$$\omega = \omega_0\,\mathbf{t} + \omega_1\,\mathbf{f}_1 + \omega_2\,\mathbf{f}_2. \tag{21}$$

Here one computes

$$\mathbf{f_1}' = \omega_0\,\mathbf{f}_2 - \omega_2\,\mathbf{t}, \tag{22}$$

which implies that $\omega_0 = \mathbf{f}_2 \cdot \mathbf{f}_1' = -\mathbf{f}_1 \cdot \mathbf{f}_2'$. Similarly the other components are obtained to be $\omega_1 = \mathbf{t} \cdot \mathbf{f}_2' = -\mathbf{f}_2 \cdot \mathbf{t}'$ and $\omega_2 = \mathbf{f}_1 \cdot \mathbf{t}' = -\mathbf{t} \cdot \mathbf{f}_1'$.

The characteristic property of an RMF is that its angular velocity has no component along $\mathbf{t}$, i.e., $\omega \cdot \mathbf{t} \equiv 0$. Equivalently, another definition of an RMF is [8] that

$$\mathbf{f}_1' \cdot \mathbf{f}_2 \equiv 0, \tag{23}$$

is a necessary-and-sufficient condition for the frame to be rotation minimizing. Then an RMF satisfies the equations

$$\begin{aligned} \mathbf{t}' &= \sigma(\omega_2\,\mathbf{f}_1 - \omega_1\,\mathbf{f}_2), \\ \mathbf{f}_1' &= -\,\sigma\,\omega_2\,\mathbf{t}, \\ \mathbf{f}_2' &= \quad \sigma\,\omega_1\,\mathbf{t}. \end{aligned} \tag{24}$$

Here the pair $\mathbf{f}_1$ and $\mathbf{f}_2$ is not unique; there exist a one-parameter family of RMFs corresponding to different sets of initial positions of $\mathbf{f}_1$ and $\mathbf{f}_2$.

2.4 RRMF Curves

For computational purposes, it is desired that an adapted frame is rational, such as a *rational rotation minimizing frame* (RRMF). Whereas not every PH curve admits an RRMF. For instance, it is shown by Han [8] that a cubic PH curve does not admit an RRMF. It is also shown in [8] that a PH curve (3) admits an RRMF if and only if there exist relatively prime polynomials $a(t)$ and $b(t)$ with gcd$(a(t), b(t)) =$ constant such that the components u, v, p, q of $\mathcal{A}$ satisfy

$$\frac{u\,v' - u'\,v - p\,q' + p'\,q}{u^2 + v^2 + p^2 + q^2} = \frac{a\,b' - a'\,b}{a^2 + b^2}. \tag{25}$$

For a quintic PH curves we will make use of the following recent results which give the conditions to have an RRMF.

Theorem 3. *([10]) A PH curve $\mathbf{r}(t)$ with $\mathbf{r}'(0) = (1, 0, 0)$ has an RRMF if and only if the Bernstein coefficients in (13) of the quadratic polynomials $\alpha(t)$ and $\beta(t)$ in the Hopf map representation (9) can be expressed in terms of two complex parameters ζ, η and one real parameter ξ in the form*

$$\begin{aligned} (\alpha_0, \alpha_1, \alpha_2) &= (1, \zeta, |\zeta|^2 - |\eta|^2 + \xi\,\mathrm{i}), \\ (\beta_0, \beta_1, \beta_2) &= (0, \eta, 2\,\bar{\zeta}\,\eta). \end{aligned} \tag{26}$$

Note that a more general form of this result is given in [10] but the above result is enough for our purpose.

There is a constraint on the quaternion coefficients $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$ of spatial PH quintics to have an RRMF.

Theorem 4. *([10]) A general spatial PH quintic specified by (7) and (12) has an RRMF if and only if the quaternion coefficients $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$ satisfy*

$$\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_2^* + \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^* = 2\,\mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_1^*. \tag{27}$$

3 RRMFs on Helices of any Degree

We aim to give results on the nonexistence of RRMFs on quintic helices, but we first give a result for curves of any degree whose proof is similar to the one in [8] to obtain the equation (25).

Lemma 1. *Let a PH curve $\mathbf{r}(t)$ given by (3) be a helical curve with $\kappa/\tau = c$ and $c \in \mathbb{R}$. Then $\mathbf{r}(t)$ has an RRMF if and only if there exist relatively prime polynomials $a(t)$ and $b(t)$ satisfying*

$$\frac{\sqrt{\rho}}{c\,\sigma} = \frac{a\,b' - a'\,b}{a^2 + b^2}, \tag{28}$$

where

$$\begin{aligned} \rho = &(u\,p' - u'\,p + v\,q' - v'\,q)^2 \\ &+(u\,q' - u'\,q + v\,p' - v'\,p)^2. \end{aligned} \tag{29}$$

Proof. Suppose that $\{\mathbf{t}, \mathbf{f}_1, \mathbf{f}_2\}$ is a frame adapted to $\mathbf{r}(t)$. As $\mathbf{r}(t)$ is a helix, its Serret-Frenet frame is rational [16]. Then, one has

$$\begin{aligned} \mathbf{f}_1 &= \frac{a^2 - b^2}{a^2 + b^2}\,\mathbf{n} - \frac{2\,a\,b}{a^2 + b^2}\,\mathbf{b}, \\ \mathbf{f}_2 &= \frac{a^2 - b^2}{a^2 + b^2}\,\mathbf{b} + \frac{2\,a\,b}{a^2 + b^2}\,\mathbf{n}, \end{aligned} \tag{30}$$

for some relatively prime polynomials $a(t)$ and $b(t)$, where $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is the Serret-Frenet frame. If $\{\mathbf{t}, \mathbf{f}_1, \mathbf{f}_2\}$ is an RRMF then it also satisfies (23). Hence by using the equations (23) and (30), and considering the Serret-Frenet equations (18) one obtains

$$\begin{aligned} \tau\,\sigma = &\left(\frac{a^2 - b^2}{a^2 + b^2}\right)\left(\frac{2\,a\,b}{a^2 + b^2}\right)' \\ &-\left(\frac{a^2 - b^2}{a^2 + b^2}\right)'\left(\frac{2\,a\,b}{a^2 + b^2}\right). \end{aligned} \tag{31}$$

After some algebra it is obtained that

$$\tau\,\sigma = 2\,\frac{a\,b' - a'\,b}{a^2 + b^2}. \tag{32}$$

On the other hand, for the curvature of a PH curve

$$\kappa = 2\,\frac{\sqrt{\rho}}{\sigma^2} \tag{33}$$

is satisfied [1]. Also we know that

$$\frac{\kappa}{\tau} = c \tag{34}$$

for some $c \in \mathbb{R}$. Then combining the equations (32), (33) and (34) gives the equation (28).

Conversely, suppose that the equation (28) is satisfied. Then by the equations (30-34) $\{\mathbf{t}, \mathbf{f}_1, \mathbf{f}_2\}$ forms an RRMF. □

The equation (28) may be useful for further results on existence or nonexistence of RRMFs on polynomial helices, nevertheless we follow the methods of [10] and [12] for the quintic case in the next section.

4 Nonexistence of RRMFs on Quintic Helices

Here we give one of our nonexistence results on helical PH quintics.

Theorem 5. *If a (regular) general helical PH quintic satisfying (15) has an RRMF, then it is a planar curve.*

To prove Theorem 5 we need the following basic result for which we supply a proof for completeness.

Lemma 2. *A polynomial space curve $\mathbf{r}(t)$ is a planar curve if*

$$\mathbf{r}'(t) = f(t)\,\mathbf{u} + g(t)\,\mathbf{v} \tag{35}$$

for two polynomials $f(t), g(t)$ and two constant vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

Proof. Differentiating the equation (35) gives that three vectors $\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)$ lie on the plane spanned by two constant vectors $\mathbf{u}$ and $\mathbf{v}$. Then by the equation (19) the torsion of $\mathbf{r}(t)$ is identically vanishing which concludes the proof. □

Proof of Theorem 5. Let $\mathbf{r}(t)$ be a quintic PH curve satisfying (15). If $\mathbf{r}(t)$ has an RRMF then the quaternion coefficients $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$ satisfy (27). Substituting (15) into (27) one computes

$$\begin{aligned} &\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_2^* + \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^* = \\ &= 2\,(c_0\,\mathcal{A}_0 + c_2\,\mathcal{A}_2)\,\mathbf{i}\,(c_0\,\mathcal{A}_0^* + c_2\,\mathcal{A}_2^*), \\ &= \frac{2\,c_0^2}{1-2\,c_0\,c_2}\,\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_0^* + \frac{2\,c_2^2}{1-2\,c_0\,c_2}\,\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_2^*, \end{aligned} \tag{36}$$

which gives $\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_2^* + \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^*$ as a linear combination of $\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_0^*$ and $\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_2^*$. By using this fact and by the equations (7), (12) and (15) we have $\mathbf{r}'(t)$ as

$$\begin{aligned} &\mathcal{A}\,\mathbf{i}\,\mathcal{A}^* = ((1-t)^2\mathcal{A}_0 + 2\,t(1-t)\mathcal{A}_1 \\ &+ t^2\mathcal{A}_2)\,\mathbf{i}\,((1-t)^2\mathcal{A}_0^* + 2\,t(1-t)\mathcal{A}_1^* + t^2\mathcal{A}_2^*). \end{aligned} \tag{37}$$

Then if we rearrange the right hand side, the terms without $\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_0^*$ and $\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_2^*$ are obtained to be

$$\begin{aligned} &(2\,t(1-t)^3c_2 + (4\,c_0\,c_2+1)\,t^2\,(1-t)^2 \\ &+2\,t^3(1-t)c_0)(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_2^* + \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^*), \end{aligned} \tag{38}$$

which is also a linear combination of the same two constant vectors $\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_0^*$ and $\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_2^*$, implying that $\mathbf{r}(t)$ is planar by Lemma 2. □

For monotone-helical quintics we have a nonexistence result of RRMFs.

Theorem 6. *There is not an RRMF on a (regular) monotone-helical PH quintic that is not a straight line.*

Proof. One can transform the curve to a special form by rotations, translations and a scaling into a curve which satisfies $\mathbf{r}'(0) = (1,0,0)$. It is known by [17] that RMFs are preserved by Möbius transformations, such as rotations and translations, so working with a curve with $\mathbf{r}'(0) = (1,0,0)$ is enough for the general. Now we can employ Theorems 2 and 3 together. Namely, if we use the property to be a monotone-helical quintic from Theorem 2, we know that $a(t) = a_0(1-t) + a_1\,t$, $b(t) = t$, and $h(t) = h_0(1-t) + h_1\,t$, where $a_0, a_1, h_0\; h_1$ are complex numbers. This gives us

$$\begin{aligned} \alpha(t) = \; &h(t)\,a(t) = a_0h_0(1-t)^2 \\ &+ (a_1h_0 + a_0h_1)(1-t)t + a_1h_1t^2. \end{aligned} \tag{39}$$

Similarly for complex polynomial $\beta(t)$, we have

$$\beta(t) = h(t)\,b(t) = h_0\,t\,(1-t) + h_1\,t^2. \tag{40}$$

After comparison between equation (13) and these expressions, it is easily seen that the equations

$$\begin{aligned} \alpha_0 &= a_0\,h_0, \quad 2\,\alpha_1 = a_1\,h_0 + a_0\,h_1, \\ \alpha_2 &= a_1\,h_1, \quad \beta_0 = 0, \\ 2\beta_1 &= h_0, \quad \beta_2 = h_1. \end{aligned} \tag{41}$$

hold. From Theorem 3, we have further restrictions,

$$\begin{aligned} a_0h_0 &= 1, \quad a_1\,h_0 + a_0\,h_1 = 2\,\zeta, \\ a_1\,h_1 &= |\zeta|^2 - |\eta|^2 + \xi\,\mathrm{i}, \quad \beta_0 = 0, \\ h_0 &= 2\,\eta, \quad h_1 = 2\,\bar{\zeta}\,\eta. \end{aligned} \tag{42}$$

Here ζ, η are complex numbers and ξ is a real number. Finally, the equations (42) should be consistent with each other. We can choose the following three equations,

$$\begin{aligned} a_1\,h_1 &= |\zeta|^2 - |\eta|^2 + \xi\,\mathrm{i}, \\ a_1 &= \frac{2\,\zeta - \bar{\zeta}}{2\,\eta}, \quad h_1 = 2\,\bar{\zeta}\,\eta. \end{aligned} \tag{43}$$

Then, we obtain

$$2\,\zeta\,\bar{\zeta} - \bar{\zeta}\,\bar{\zeta} = |\zeta|^2 - |\eta|^2 + \xi\,\mathrm{i}. \tag{44}$$

If take $\zeta = \zeta_1 + \zeta_2\,\mathrm{i}$, we get

$$2\,\zeta_2^2 + 2\,\zeta_1\,\zeta_2\,\mathrm{i} = -|\eta|^2 + \xi\,\mathrm{i}. \tag{45}$$

The calculation gives us $\eta = 0$ which contradicts with equations (42). Recall that $\eta = 0$ implies that $\mathbf{r}(t)$ is a straight line [10]. □

5 Conclusion

In this paper, a characterization of regular quintic helical PH curves that allow RRMFs has been given. We want to point out here that the formula given in Lemma 1 suggests a simplification of rational approximation to RMFs on monotone-helical quintic PH curves, which will be reported in a future paper. As the main result, we have shown that quintic helices cannot have RRMFs. Although we do not have a rigorous proof we predict that there are not RRMFs on seven degree PH helices. We can see this fact from Theorem 1 in [3] which gives a condition for a polynomial curve to be a helix and an equation giving a condition for a curve to have RRMFs in the paper [11]. Therefore if there is not an RRMF on a curve an approximation of RRMFs can be done as in [18]. We plan to develop existing rational approximation methods to RMFs for especially quintic helices and apply it to sweep surface modeling and rigid body design.

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