Rotation minimizing frames on monotone-helical PH quintics: approximation and applications to modeling problems

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Abstract

A *rotation minimizing frame* (RMF) $\{\mathbf{t}, \mathbf{f}_1, \mathbf{f}_2\}$ of a curve in 3-space consists of the tangent $\mathbf{t}$ and two normal vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ which rotate as little as possible around $\mathbf{t}$. Having the property of minimum twist makes RMFs attractive in computer graphics, swept surface constructions, motion design and similar applications. Recently we have shown that there is not any rational RMF (RRMF) on monotone quintic helices, so this motivates to develop a rational approximation to RMFs. It is shown that rational approximation to RMFs on monotone-helical Pythagorean-hodograph (PH) quintics is computationally cheap, then it is applied to profile surface modeling and rigid body design.

1 Introduction

1.1 General context

A parametric curve $r(t) = (x(t), y(t), z(t))$ is called a *Pythagorean-hodograph* (PH) curve if its speed is a polynomial [2]. The theory of PH curves is a much studied research topic in Computer Aided Geometric Design (CAGD) because of their useful properties. An *adapted frame* on a space curve $\mathbf{r}(t)$ is an orthonormal moving frame $\{\mathbf{t}, \mathbf{f}_1, \mathbf{f}_2\}$ such that, $\mathbf{t}$ is the unit tangent $\mathbf{r}'(t)/|\mathbf{r}'(t)|$, and the other two vectors span the normal plane. Rotation minimizing frames (RMFs) have minimum twist that makes them distinguished among adapted frames. RMFs are used in animation, robotics applications, the construction of swept surfaces [10] where the axis of a tool should remain tangential to a given spatial path while minimizing changes of orientation about this axis.

1.2 Motivation

It is easy to compute *exact* derivation of RMFs on spatial PH curves [2]. Further, in practical applications, especially *rational* RMFs (RRMFs) are useful for computational purposes. The only curves having rational adapted frames are PH curves, since the only PH curves have rational unit tangent vectors. Besides, the arc lengths of PH curves can be computed precisely, and it can formulate real-time interpolators to drive multi-axis CNC machines along curve paths, at fixed or varying speeds from their exact analytic descriptions [6].

In the family of PH curves, polynomial helices have remarkable interest, particularly quintic helices. The relationship between such curves and some problems in the realm of computer-aided design of curves and surfaces show that the suitable curves are helical PH quintics for real applications [3].

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1.3 Problem Statement

Having observed that in general PH quintic helices cannot have RRMFs, we aim at making rational approximations to RMFs. We focus on *monotone-helical PH quintics*, i.e. curves whose hodograph has coordinates with a common factor, say h. On a monotone-helical PH quintic curve $\mathbf{r}(t)$, RMFs can be computed easily since there is a simplification in the integral giving the angle θ between *Frenet-Serret frame* (FSF) and RMFs. Because, θ is given by

$$\theta(t) - \theta_0 = -\int \tau\, \sigma\, dt, \qquad (1)$$

where $\sigma = |\mathbf{r}'(t)|$ and τ is the torsion, and for monotone-helical PH quintics the integrand $\tau\,\sigma$ turns out to be $\frac{2}{gc}$, with $\sigma = h\,g$ and c is the helicity constant. Applying this idea to related topics, such as sweep surface modeling and rigid body motion design, is the subject of this work.

1.4 Related Work

In the previous work [11], we showed that there does not exist RRMFs on monotone-helical PH quintics.

Theorem 1 *[11] There is not an RRMF on a (regular) monotone-helical PH quintic that is not a straight line.*

We also gave a condition (9) on a polynomial helix of any degree to have an RRMF. This condition leads to a simplification of rational approximation to RMFs on monotone-helical PH quintics. For PH cubic curves and more generally PH curves rational approximation to RMFs was studied in [5, 9].

This is an extended abstract of a presentation given at EuroCG 2013. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear in a conference with formal proceedings and/or in a journal.

1.5 Overview of the Results

The present paper is organized as follows. Section 2 introduces definitions of and basic results on monotone-helical PH quintics, RMFs, RRMFs, and profile surfaces. In Section 3 we discuss a minimax rational approximation to an RMF on a monotone-helical PH quintic. Then we discuss applications to sweep surface modeling in the same section and to rigid body motion planning in Section 4. Subsequently in Section 5 we conclude with remarks about our future considerations.

2 Background

In this section we review the preliminary material which we use along the paper.

2.1 Monotone-Helical PH Quintics

In Hopf map $\mathbb{C} \times \mathbb{C} \to \mathbb{R}^3$ representation, a PH curve $\mathbf{r}(t)$ is defined by its hodograph

$$\mathbf{r}'(t) = (2\,\alpha(t)\,\bar{\beta}(t), |\alpha(t)|^2 - |\beta(t)|^2), \qquad (2)$$

where

$$\alpha(t) = u(t) + v(t)\,\mathrm{i} \text{ and } \beta(t) = q(t) + p(t)\,\mathrm{i}, \qquad (3)$$

are some complex polynomials, and the identification $\mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$ is assumed [2].

A *monotone-helical PH curve* $\mathbf{r}(t)$ is a quintic PH curve whose hodograph have components with a common quadratic factor, then $\alpha(t) = h(t)\,a(t)$ and $\beta(t) = h(t)\,b(t)$ for linear complex polynomials $a(t), b(t)$ and $h(t)$. Then the helical PH quintic curve (2) is given by

$$\mathbf{r}'(t) = |h(t)|^2 (2\,a(t)\,\bar{b}(t), |a(t)|^2 - |b(t)|^2). \qquad (4)$$

Example [2]: Let us consider the monotone-helical PH quintic curve $\mathbf{r}(t) = (x(t), y(t), z(t))$, where

$$\begin{aligned} u(t) &= t^2 - 3\,t, & v(t) &= t^2 - 5\,t + 10, \\ p(t) &= -\,2\,t^2 + 3\,t + 5, & q(t) &= t^2 - 9\,t + 10. \end{aligned} \qquad (5)$$

Here a common factor of the components $x(t)$, $y(t)$, $z(t)$ is $h(t) = t^2 - 2\,t + 5$. The helicity constant is obtained to be $c = \kappa/\tau = 5\sqrt{2}/3$, where κ, τ are the curvature and torsion, respectively. We will make use of this curve to demonstrate our approximation results.

2.2 Rotation Minimizing Frames

The most canonical adapted frame is the FSF $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$. There are many other adapted frames associated with a given space curve $\mathbf{r}(t)$, and among them the RMFs are the ones which minimize the amount of rotation along the curve. The variation of a frame $\{\mathbf{t}, \mathbf{f}_1, \mathbf{f}_2\}$ defined on a curve $\mathbf{r}(t)$ is given by its vector angular velocity $\omega = \omega_0\,\mathbf{t} + \omega_1\,\mathbf{f}_1 + \omega_2\,\mathbf{f}_2$ with the relations

$$\mathbf{t}' = \omega \times \mathbf{t}, \quad \mathbf{f}_1' = \omega \times \mathbf{f}_1, \quad \mathbf{f}_2' = \omega \times \mathbf{f}_2. \qquad (6)$$

The characteristic property of an RMF is that its angular velocity has no component along $\mathbf{t}$, i.e.,

$$\omega \cdot \mathbf{t} \equiv 0. \qquad (7)$$

As we consider a helix $\mathbf{r}(t)$, its FSF is rational [2]. Observe that an RMF is given by a rotation in the normal plane

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix} = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \end{pmatrix}, \qquad (8)$$

where (1) with the integration constant θ_0 [2]. Therefore an RMF is not rational in general.

2.3 Rational Frames of Quintic PH Helices

A general condition on helices of any degree to have RRMFs is also given in [11].

Lemma 2 *[11] Let a PH curve $\mathbf{r}(t)$ be a helical curve with $\kappa/\tau = c$ and $c \in \mathbb{R}$. Then $\mathbf{r}(t)$ has an RRMF if and only if there exist relatively prime polynomials $\mu(t)$ and $\nu(t)$ satisfying*

$$\frac{\sqrt{\rho}}{c\,\sigma} = \frac{\mu\,\nu' - \mu'\,\nu}{\mu^2 + \nu^2}, \qquad (9)$$

where

$$\begin{aligned} \rho = &(u\,p' - u'\,p + v\,q' - v'\,q)^2 + \\ &(u\,q' - u'\,q + v\,p' - v'\,p)^2. \end{aligned} \qquad (10)$$

The proof of Lemma 2 gives an idea of a simplification of rational approximation to RMFs on monotone-helical PH quintic curves. It will be detailed in the next section.

2.4 Profile Surfaces

A *profile surface* is a sweep surface generated by an RMF. More explicitly, it has a parametric representation

$$\mathbf{S}(s,t) = \mathbf{r}(t) + \mathbf{f}_1(t)\,c_1(s) + \mathbf{f}_2(t)\,c_2(s), \qquad (11)$$

where $\mathbf{r}(t)$ is the *spine curve* with parameter $t \in [t_0, t_1] \in \mathbb{R}$, $c(s) = (c_1(s), c_2(s))^T$ is the *cross section* or *profile curve* with parameter $s \in [s_0, s_1] \subset \mathbb{R}$, and $\{\mathbf{t}, \mathbf{f}_1, \mathbf{f}_2\}$ is an RMF along $\mathbf{r}(t)$.

If the cross-section curve is a straight line, then the profile surface is a developable surface [9]. This implies that they are flat surfaces, i.e. they have vanishing Gauss curvature $K = 0$. In the next section we obtain a rational approximation of an RMF on a monotone-helical PH quintics.

3 Minimax Rational Approximation on monotone-helical PH quintics

In this section we will make a minimax rational approximation on monotone-helical PH quintics by using Mathematica. A (m, k) degree rational function is the ratio of a degree m polynomial to a degree k polynomial. The *error* of minimax rational approximation is the difference between the function and its approximation w.r.t. Euclidean norm. The aim of minimax rational approximation is to minimize the maximum of the relative error from the polynomial curve.

Let $f(t)$ be continuous on a closed interval $[t_0, t_1]$. Then there exists a unique (m, k) degree rational polynomial $\frac{a(t)}{b(t)}$, called the *minimax rational approximation* to exact function $f(t)$, that minimizes

$$\varepsilon(a(t), b(t)) = \max_{t_0<t<t_1} \mid f(t) - \tfrac{a(t)}{b(t)} \mid . \qquad (12)$$

3.1 Minimax Rational Approximation of RMFs on Monotone-Helical PH Quintics

Nonexistence of RRMFs on a monotone-helical PH quintic curve motivates an approximation of RRMFs which can be done as in [5] with further simplifications as indicated in the following. Standard parametrization of circle is

$$(\sin\theta, \cos\theta) = \left(\frac{2f}{1+f^2}, \frac{1-f^2}{1+f^2}\right), \qquad (13)$$

where $f = \tan\frac{\theta}{2}$. Then, one can make a rational approximation by

$$f(t) = \tan\frac{\theta(t)}{2} = -\tan\left(\int \frac{\tau\,\sigma}{2}\,dt\right) \simeq \frac{a(t)}{b(t)}, \quad (14)$$

for some relatively prime polynomials $a(t)$ and $b(t)$, which gives a rational frame

$$\begin{pmatrix}\tilde{\mathbf{f}}_1\\ \tilde{\mathbf{f}}_2\end{pmatrix} = -\frac{1}{a^2+b^2}\begin{pmatrix}a^2-b^2 & -2\,a\,b\\ 2\,a\,b & a^2-b^2\end{pmatrix}\begin{pmatrix}\mathbf{n}\\ \mathbf{b}\end{pmatrix}. \quad (15)$$

For a quintic helix, the integrand $\tau\,\sigma$ is a rational function of degree $(2, 4)$, while for monotone-helical PH quintic curves this simplifies to $(0, 2)$. This is because the monotone-helical PH curve identities hold:

$$\sigma = h\,g \quad \text{and} \quad \rho = h^2, \qquad (16)$$

where $h = gcd(x', y', z')$ [4]. Furthermore we put together the curvature of a PH curve [2] $\kappa = 2\,\frac{\sqrt{\rho}}{\sigma^2}$, helicity condition $c = \kappa/\tau$, and monotone-helical conditions (16) in the following computation:

$$\tau\,\sigma = \frac{\kappa}{c}\,\sigma = 2\,\frac{\sqrt{\rho}}{c\,\sigma^2}\,\sigma = \frac{2}{g\,c}. \qquad (17)$$

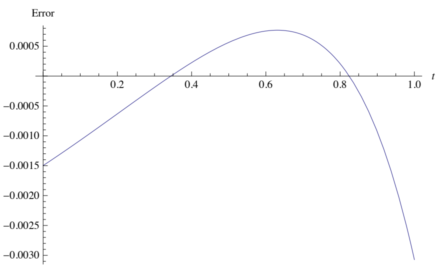

Figure 1: Error for the RMF condition (7).

Then from equation (1), we get

$$\frac{\theta}{2} = -\int \frac{1}{c\,g}\,dt. \qquad (18)$$

The integral in (18) gives a simplification of computations when making the approximation given in (14).

Example: Let us consider the monotone-helical PH quintic curve (5). After applying the minimax rational approximation with error $\varepsilon = -0.000172464$, the rational approximation to exact function $f(t)$ is then

$$\frac{a(t)}{b(t)} = \frac{-0.469264 + 0.225286\,t}{1 - 0.304786\,t}. \qquad (19)$$

This result yields a good RRMF approximant as can be seen by Figure 1 which shows the error of the RMF condition (7).

3.2 Minimax Rational Approximation of Profile Surfaces

Rational approximation of RMF can be used to generate profile surfaces with rational representation. If the profile curve $c(s)$ is chosen to be a straight line then the rational approximation to the profile surface is expected to have Gauss curvature close to zero values.

Example: Consider two sweep surfaces,

$$\begin{aligned}\mathbf{S}_1(s,t) &= \mathbf{r}(t) + (-\tfrac{1}{5}\,s+5)\,\tilde{\mathbf{f}}_1 + (10\,s - \tfrac{1}{2})\,\tilde{\mathbf{f}}_2,\\ \mathbf{S}_2(s,t) &= \mathbf{r}(t) + (-\tfrac{1}{5}\,s+5)\,\mathbf{n} + (10\,s - \tfrac{1}{2})\,\mathbf{b},\end{aligned} \qquad (20)$$

generated by the rational approximation to the RMF (left) and by the FSF (right) of the monotone-helical PH quintic given in (5), see Figure 2. The Gaussian curvature $\tilde{K}$ can be used as an accuracy criterion. Since the cross-section curve

$$c(s) = (-\tfrac{1}{5}\,s+5, 10\,s - \tfrac{1}{2})^T \qquad (21)$$

in this example is a straight line, the Gauss curvature of a profile surface is vanishing. For profile surface $\mathbf{S}_1(s,t)$, minimum and maximum values of the Gauss curvature are

$$\begin{aligned}\tilde{K}_{min}(0.899997, 1) &= -\,9.76359 \times 10^{-11},\\ \tilde{K}_{max}(0.899992, 14.7928) &= -\,1.11723 \times 10^{-11}.\end{aligned} \qquad (22)$$

Figure 2: Sweep surfaces $\mathbf{S}_1(s,t)$ and $\mathbf{S}_2(s,t)$, generated by the rational approximation to the RMF (left) and by the FSF (right).

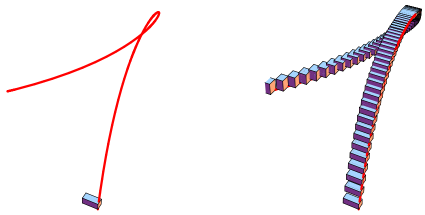

Figure 3: Rigid Body with initial configuration on a monotone-helical PH quintic curve $\mathbf{r}(t)$ (left) and the same curve with rigid body motion (right).

Our approximation $\tilde{K}$ is between the values $\tilde{K}_{min}$ and $\tilde{K}_{max}$ which are close to zero. Therefore this criterion shows us that our approximation gives good results.

4 Rigid Body Motion Design

A rigid body motion can be modeled as the motion of an adapted frame. As they make minimum twist, RMFs are very useful in rigid body motion design, however computation of these frames requires to integrate complicates functions. For PH curves the integral in (1) is known to be integrated by elementary functions [2]. We can further see by (18) that this integration is very useful from a computational point of view. This considerable feature of monotone-helical PH quintics can be employed in the following.

Assume that an initial point p_0 and a final point p_1, and an initial frame at p_0 are given. We illustrate that rigid body motion design problem in Figure 3. To find a trajectory satisfying these initial data, a monotone-helical PH quintic can be obtained under some more suitable conditions, then it is an easy task to compute an RMF which aligns with the initial frame at p_0. For this purpose interpolation method for monotone-helical PH quintics given in [7] can be used.

5 Conclusions and Future Work

Rational approximation of RMFs on monotone-helical PH quintics is studied. It is observed in this work that the integrand (1) which is used to compute RMFs is a rational function of degree (0,2). This leads to a simplification in rational approximation to RMFs as we touch upon in this work. Moreover, it is pointed out that several applications can be done to modeling problems such as sweep surfaces and rigid body design. It is worth to mention here that this distinctive feature is special for monotone-helical PH quintics.

Future work will be to improve and apply the observations outlined above. One concrete question arising is the following. When we are modeling rigid body motion, there exist singular points on the monotone curve. We will search for a method to remove these singularities of monotone-helical PH quintics.

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